APPENDIX A MATHEMATICAL REVIEW

This appendix presents a simple review of the mathematical tools used throughout the book. It assumes some basic familiarity with calculus and covers techniques that are commonly used in modeling economic growth and development. A special effort has been made to include more words than equations. We hope this will permit a quick and easy understanding of the mathematics used in this book. For additional details, please refer to an introductory calculus textbook.

A.1 DERIVATIVES

The derivative of some function $f(x)$ with respect to x reveals how $f(\cdot)$ changes when x changes by a very small amount. If $f(\cdot)$ increases when x increases, then $df/dx > 0$, and vice versa. For example, if $f(x) = 5x$, then $df/dx = 5$, or $df = 5\ dx$: For every small change in x, $f(\cdot)$ changes by 5 times that amount.

A.1.1 WHAT DOES $\dot{K}$ MEAN?

In discussing economic growth, the most common derivative used is a derivative with respect to time. For example, the capital stock, K, is a function of time t, just like f was a function of x above. We can ask how the capital stock changes over time; this is fundamentally a question about the derivative dK/dt. If the capital stock is growing over time, then $dK/dt > 0$.

For derivatives with respect to time, it is conventional to use the "dot notation": dK/dt is then written as $\dot{K}$ — the two expressions are equivalent. For example, if $\dot{K} = 5$, then for each unit of time that passes, the capital stock increases by 5 units.

Notice that this derivative, $\dot{K}$, is very closely related to $K_{1997} - K_{1996}$. How does it differ? First, let's rewrite the change from 1996 to 1997 as $K_t - K_{t-1}$. This second expression is more general; we can evaluate it at $t = 1997$ or at $t = 1990$ or at $t = 1970$. Thus we can think of this change as a change per unit of time, where the unit of time is one period. Next, $\dot{K}$ is an *instantaneous* change rather than the change across an entire year. We could imagine calculating the change of the capital stock across one year, or across one quarter, or across one week, or across one day, or across one hour. *As the time interval across which we calculate the change shrinks, the expression $K_t - K_{t-1}$, expressed per unit of time, approaches the instantaneous change $\dot{K}$.* Formally, this is exactly the definition of a derivative. Let Δt be our time interval (a year, a day, or an hour). Then,

$$\lim_{\Delta t \to 0} \frac{K_t - K_{t-\Delta t}}{\Delta t} = \frac{dK}{dt}.$$

A.1.2 WHAT IS A GROWTH RATE?

Growth rates are used throughout economics, science, and finance. In economics, examples of growth rates include the inflation rate — if the inflation rate is 3 percent, then the price level is rising by 3 percent per year. The population growth rate is another example — population is increasing at something like 1 percent per year in the advanced economies of the world.

The easiest way to think about growth rates is as percentage changes. If the capital stock grew by 4 percent last year, then the change in the capital stock over the course of the last year was equal to 4 percent of its starting level. For example, if the capital stock began at $10 trillion and rose to $10.4 trillion, we might say that it grew by 4 percent. So one way of calculating a growth rate is as a percentage change:

$$\frac{K_t - K_{t-1}}{K_{t-1}}.$$

For mathematical reasons that we will explore below, it turns out to be easier in much of economics to think about the *instantaneous* growth rate. That is, we define the growth rate to be the derivative dK/dt divided by its starting value, K. As discussed in the preceding section, we use $\dot{K}$ to represent dK/dt. Therefore, $\dot{K}/K$ is a growth rate. Whenever you see such a term, just think "*percentage change.*"

A few examples may help clarify this concept. First, suppose $\dot{K}/K = .05$; this says that the capital stock is growing at 5 percent per year. Second, suppose $\dot{L}/L = .01$; this says that the labor force is growing at 1 percent per year.

A.1.3 GROWTH RATES AND NATURAL LOGS

The mathematical reason why this definition of growth rates is convenient can be seen by considering several properties of the natural logarithm:

1. If $z = xy$, then $\log z = \log x + \log y$.
2. If $z = x/y$, then $\log z = \log x - \log y$.
3. If $z = x^{\beta}$, then $\log z = \beta \log x$.
4. If $y = f(x) = \log x$, then $dy/dx = 1/x$.
5. If $y(t) = \log x(t)$, then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{x}\dot{x} = \frac{\dot{x}}{x}.$$

The first of these properties is that the natural log of the product of two (or more) variables is the sum of the logs of the variables. The second property is very similar, but relates the division of two variables to the difference of the logs. The third property allows us to convert exponents into multiplicative terms. The fourth property says that the derivative of the log of some variable x is just $1/x$.

The fifth property is a key one. In effect, it says that *the derivative with respect to time of the log of some variable is the growth rate of* *that variable*. For example, consider the capital stock, K. According to property 5 above,

$$\frac{d \log K}{dt} = \frac{\dot{K}}{K},$$

which, as we saw in Section A.1.3, is the growth rate of K.

A.1.4 "TAKE LOGS AND DERIVATIVES"

Each of the properties of the natural logarithm listed in the preceding section is used in the "take logs and derivatives" example below. Consider a simple Cobb-Douglas production function:

$$Y = K^{\alpha} L^{1-\alpha}.$$

If we take logs of both sides,

$$\log Y = \log K^{\alpha} + \log L^{1-\alpha}.$$

Moreover, by property 3 discussed in section A.1.3,

$$\log Y = \alpha \log K + (1-\alpha) \log L.$$

Finally, by taking derivatives of both sides with respect to time, we can see how the growth rate of output is related to the growth rate of the inputs in this example:

$$\frac{d \log Y}{dt} = \alpha \frac{d \log K}{dt} + (1-\alpha) \frac{d \log L}{dt},$$

which implies that

$$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1-\alpha) \frac{\dot{L}}{L}.$$

This last equation says that the growth rate of output is a weighted average of the growth rates of capital and labor.

A.1.5 RATIOS AND GROWTH RATES

Another very useful application of these properties is in situations in which the ratio of two variables is constant. First, notice that if a variable is constant, its growth rate is zero — it is not changing, so its time derivative is zero.

Now, suppose that $z = x/y$ and suppose we know that z is constant over time—i.e., $\dot{z} = 0$. Taking logs and derivatives of this relationship, one can see that

$$\frac{\dot{z}}{z} = \frac{\dot{x}}{x} - \frac{\dot{y}}{y} = 0 \Longrightarrow \frac{\dot{x}}{x} = \frac{\dot{y}}{y}.$$

Therefore, if the ratio of two variables is constant, the growth rates of those two variables must be the same. Intuitively, this makes sense. If the numerator of the ratio were growing faster than the denominator, the ratio itself would have to be growing over time.

A.1.6 Δlog VERSUS PERCENTAGE CHANGE

Suppose a variable exhibits exponential growth:

$$y(t) = y_0 e^{gt}.$$

For example, $y(t)$ could measure per capita output for an economy. Then,

$$\log y(t) = \log y_0 + gt,$$

and therefore the growth rate, g, can be calculated as

$$g = \frac{1}{t}(\log y(t) - \log y_0).$$

Or, calculating the growth rate between time t and time $t-1$,

$$g = \log y(t) - \log y(t-1) \equiv \Delta \log y(t).$$

These last two equations provide the justification for calculating growth rates as the change in the log of a variable.

How does this calculation relate to the more familiar percentage change? The answer is straightforward:

$$\begin{aligned}\frac{y(t) - y(t-1)}{y(t-1)} &= \frac{y(t)}{y(t-1)} - 1 \\ &= e^{g} - 1.\end{aligned}$$

Recall that the Taylor approximation for the exponential function is $e^x \approx 1 + x$ for small values of x. Applying this to the last equation shows that the percentage change and the change in log calculations are approximately equivalent for small growth rates:

$$\frac{y(t) - y(t-1)}{y(t-1)} \approx g.$$

A.2 INTEGRATION

Integration is the calculus equivalent of summation. For example, one could imagine a production function written as

$$Y = \sum_{i=1}^{10} x_i = x_1 + x_2 + \cdots + x_{10}, \qquad \text{(A.1)}$$

that is, output is simply the sum of ten different inputs. One could also imagine a related production function

$$Y = \int_0^{10} x_i \, di. \qquad \text{(A.2)}$$

In this production function, output is the weighted sum of a continuum of inputs x_i that are indexed by the interval of the real line between 0 and 10. Obviously, there are an infinite number of inputs in this second production function, because there are an infinite number of real numbers in this interval. However, each input is "weighted" by the average size of an interval, di, which is very small. This keeps production finite, even if each of our infinite number of inputs is used in positive amounts. Don't get too confused by this reasoning. Instead, think of integrals as sums, and think of the second production function in the same way that you would think of the first. To show you that you won't go too far wrong, suppose that 100 units of each input are used in both cases: $x_i = 100$ for all i. Output with the production function in equation (A.1) is then equal to 1,000. What is output with the production function in equation (A.2)?

$$Y = \int_0^{10} 100 di = 100 \int_0^{10} di = 1{,}000.$$

Output is the same in both cases.

A.2.1 AN IMPORTANT RULE OF INTEGRATION

In this last step we used an important rule of integration. Integrals and derivatives are like multiplication and division — they "cancel":

$$\int dx = x + C,$$

where C is some constant, and

$$\int_a^b dx = b - a.$$

A.3 SIMPLE DIFFERENTIAL EQUATIONS

There is really only one differential equation in this book that we ever need to solve: the key differential equation that relates growth rates and levels. Its solution is straightforward.

Suppose a variable x is growing at some constant rate g. That is,

$$\frac{\dot{x}}{x} = g.$$

What does this imply about the level of x? The answer can be seen by noting that the growth rate of x is the derivative of the log:

$$\frac{d \log x}{dt} = g.$$

The key to solving this differential equation is to recall that to "undo" derivatives, we use integrals. First, rewrite the differential equation slightly:

$$d \log x = g \, dt.$$

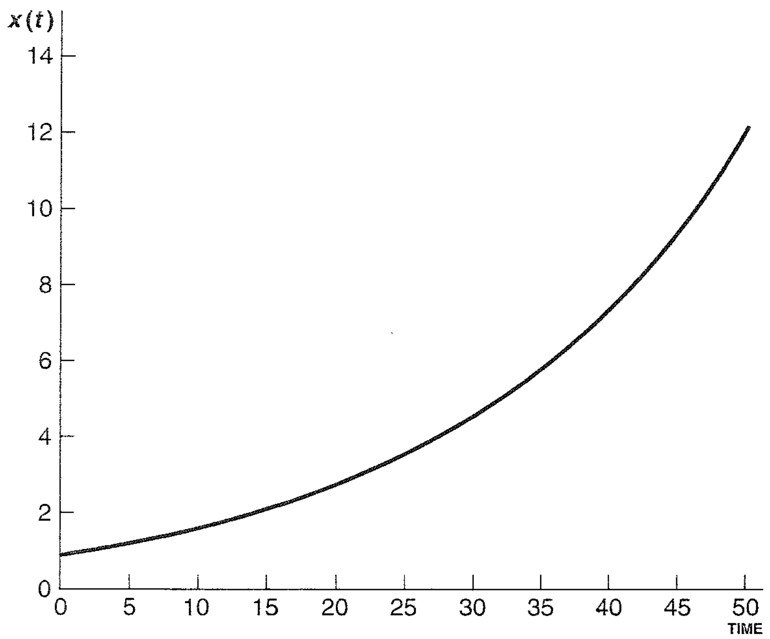
Now, integrate both sides of this equation:

$$\int d \log x = \int g \, dt,$$

which implies that

$$\log x = gt + C,$$

FIGURE A.1 **EXPONENTIAL GROWTH**



where, once again, C is some constant. Therefore, the natural logarithm of a variable that is growing at a constant rate is a linear function of time. Taking the exponential of both sides, we get

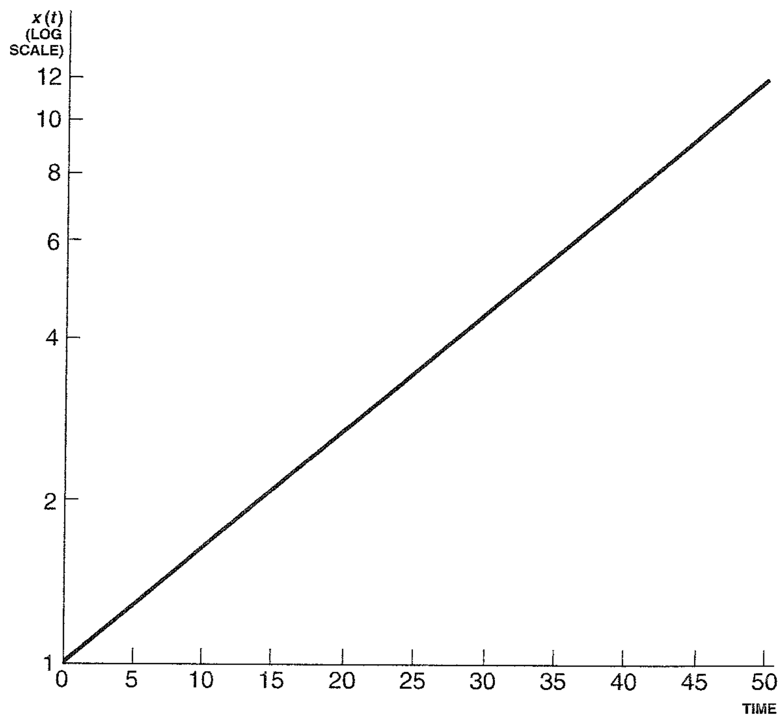
$$x = \overline{C}e^{gt}, \tag{A.3}$$

where $\overline{C}$ is another constant.[1] To figure out what the constant is, set $t = 0$ to see that $x(0) = \overline{C}$. Typically, we assume that $x(0) = x_0$, that is, at time 0, x takes on a certain value x_0. This is known as an *initial condition*. Thus $\overline{C} = x_0$. This reasoning shows why we say that a variable growing at a constant rate exhibits "exponential" growth. Figure A.1 plots $x(t)$ for $x_0 = 1$ and $g = .05$.

It is often convenient to plot variables that are growing at an exponential rate in log terms. That is, instead of plotting $x(t)$, we plot $\log x(t)$. To see why, notice that for the example we have just considered, $\log x(t)$

[1]To be exact, $\overline{C} = e^C$.

FIGURE A.2 **$x(t)$ ON A LOG SCALE**



is a linear function of time:

$$\log x(t) = \log x_0 + gt.$$

Figure A.2 plots $\log x(t)$ to show this linear relationship. Note that the slope of the relationship is the growth rate of $x(t)$, $g = .05$.

Finally, notice that it is sometimes convenient to plot the log of a variable but then to change the labels of the graph. For example, we might plot the log of per capita GDP in the U.S. economy over the last 125 years, as in Figure 1.4 in Chapter 1, to illustrate the fact that the average growth rate is fairly constant. Per capita income in 1994 was nearly $25,000. The log of 25,000 is 10.13, which is not a very informative label. Therefore, we plot the log of per capita GDP, and then relabel the point 10.13 as $25,000. Similarly, we relabel the point 8.52 as $5,000. (Why?) This relabeling is typically indicated by the statement that the variable is plotted on a "log scale."

A.3 COMPOUND INTEREST

A classic example to illustrate the difference between the "instantaneous" growth rates used in this book and the "percentage change" calculations that we are all familiar with is the difference between *continuously* compounded interest and interest that is compounded daily or yearly. Recall that interest is compounded when a bank pays you interest on your interest. (This contrasts with simple interest, where a bank pays interest only on the principal.) Suppose that you open a bank account with \$100 and the bank pays you an interest rate of 5 percent compounded yearly. Let $x(t)$ be the bank balance, and let t indicate the number of years the \$100 has been in the bank. Then, for interest compounded yearly at 5 percent, $x(t)$ behaves according to

$$x(t) = 100(1 + .05)^t.$$

The first column of Table A.1 reports the bank balance at various points in time.

Now suppose instead of being compounded yearly the interest is compounded continuously — it is not compounded every year, or every day, or every minute, but rather it is compounded every instant. As in the case of interest compounded yearly, the bank balance is growing at a rate of 5 percent. However, now that growth rate is an *instantaneous* growth rate instead of an *annual* growth rate. In this case, the bank balance obeys the differential equation $\dot{x}/x = .05$, and from the calculations we have done before leading us to equation (A.3), we know the solution to this differential equation is

$$x(t) = 100e^{.05t}.$$

TABLE A.1 BANK BALANCE WITH COMPOUND INTEREST AT 5 PERCENT

Years	Compounded yearly	Compounded continuously
0	\$100.00	\$100.00
1	105.00	105.10
2	110.20	110.50
5	127.60	128.40
10	162.90	164.90
14	198.00	201.40
25	338.60	349.00

The second column of Table A.1 reports the bank balance for this case. Notice that even after one year, the continuous compounding produces a balance slightly larger than \$105, but the differences are fairly small (at least for the first fifteen years or so).[2]

This example comparing continuously compounded interest with annually compounded interest is mathematically equivalent to comparing instantaneous growth rates of, say, output per worker to annual percentage changes in output per worker.

A.4 MAXIMIZATION OF A FUNCTION

Many problems in economics take the form of *optimization* problems: a firm maximizes profits, consumers maximize utility, etc. Mathematically, these optimization problems are solved by finding the *first-order conditions* for the problem.

For an optimization problem with only one choice variable and no constraints, the solution is particularly easy. Consider the following problem:

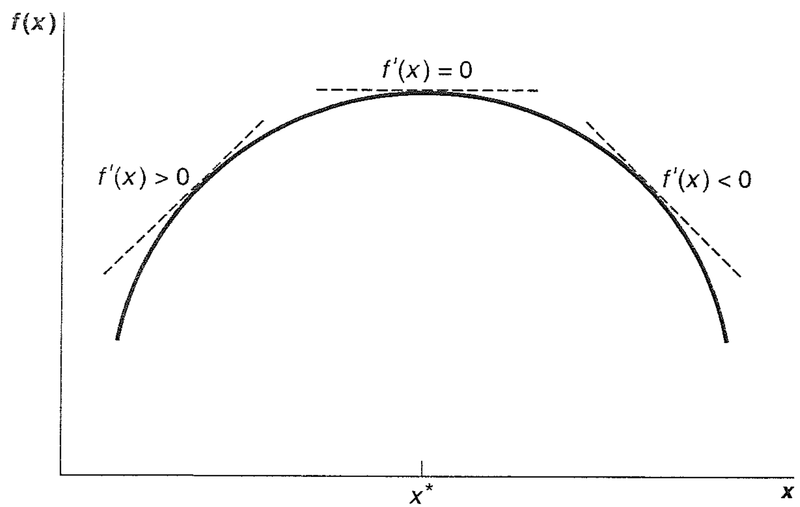
$$\max_x f(x).$$

The solution is usually found from the first-order condition that $f'(x) = 0$. Why? Suppose we guess a value x_1 for the solution and $f'(x_1) > 0$. Obviously, then, we could increase x slightly and this would increase the function. So x_1 cannot be a solution. A similar trick would work if $f'(x_1) < 0$. Therefore, the first-order condition is that the derivative, $f'(x)$, equal exactly zero at the solution.

How do we know if some point x^* that satisfies $f'(x^*) = 0$ is a maximum or a minimum (or an inflection point)? The answer involves the *second-order condition*. Figure A.3 provides the intuition behind

[2]Notice also that the \$100 doubles in about fourteen years if the interest rate is 5 percent, as predicted by the formula in Chapter 1.

FIGURE A.3 **MAXIMIZING A FUNCTION**



the second-order condition. For x^* to be a maximum, it must be the case that $f''(x^*) < 0$. That is, the first derivative must be decreasing in x at the point x^*. This way, $f'(x)$ is positive at a point just below x^* and negative at a point just above x^*. That is, $f(\cdot)$ is increasing at points below x^* and decreasing at points above x^*.

More general optimization problems with more variables and constraints follow this same kind of reasoning. For example, suppose a firm takes the wage w, the rental rate r, and the price p of its output as given and has to decide how much capital K and labor L to hire in order to produce some output:

$$\max_{K,L} \ \pi = pF(K,L) - wL - rK.$$

The first-order conditions for this problem are the familiar conditions that the wage and rental rates equal the marginal revenue product of labor and capital:

$$p\frac{\partial F}{\partial L} = w$$

and

$$p\frac{\partial F}{\partial K} = r.$$

The second-order conditions for a problem with more than one choice variable are a bit more complicated, and we will simply assume that the second-order conditions hold throughout this book (the problems are set up so that this is a valid assumption). Problems with constraints are only a bit more complicated. Refer to an intermediate microeconomics textbook for the techniques of constrained optimization. These techniques are not used in this book.

EXERCISES

1. Suppose $x(t) = e^{.05t}$ and $z(t) = e^{.01t}$. Calculate the growth rate of $y(t)$ for each of the following cases:

(a) $y = x$

(b) $y = z$

(c) $y = xz$

(d) $y = x/z$

(e) $y = x^{\beta}z^{1-\beta}$, where $\beta = 1/2$

(f) $y = (x/z)^{\beta}$, where $\beta = 1/3$.

2. Express the growth rate of y in terms of the growth rates of k, l, and m for the following cases. Assume β is some arbitrary constant.

(a) $y = k^{\beta}$

(b) $y = k/m$

(c) $y = (klm)^{\beta}$

(d) $y = (kl)^{\beta}(1/m)^{1-\beta}$.

3. Assume $\dot{x}/x = .10$ and $\dot{z}/z = .02$, and suppose that $x(0) = 2$ and $z(0) = 1$. Calculate the numerical values of $y(t)$ for $t = 0, t = 1, t = 2$, and $t = 10$ for the following cases:

(a) $y = xz$

(b) $y = z/x$

(c) $y = x^{\beta} z^{1-\beta}$, where $\beta = 1/3$.

4. Using the data from Appendix C, pp. 216 on GDP per worker in 1960 and 1997, calculate the average annual growth rate of GDP per worker for the following countries: the United States, Canada, Argentina, Chad, Brazil, and Thailand. Confirm that this matches the growth rates reported in Appendix C. (Note: Your numbers may not match exactly due to rounding error.)

5. Assuming population growth and labor force growth are the same (why wouldn't they be?), use the results from the previous exercise together with the population growth rates from Appendix C to calculate the average annual growth rate of GDP for the same group of countries.

6. On a sheet of paper (or on the computer if you'd like), make a graph with the log of GDP per worker for 1997 on the y-axis and years of schooling on the x-axis for the same countries as in Exercise 4 using the data from Appendix C. Relabel the y-axis so that it is in units of dollars per worker on a log scale.

APPENDIX B READINGS OF INTEREST

A number of very readable articles and books related to economic growth make excellent supplementary reading for students using this textbook. Some of these have been mentioned briefly in the text, others have not. This appendix gathers these references in one place.

Barro, Robert J. *Determinants of Economic Growth: A Cross-Country Empirical Study.* Cambridge, Mass.: MIT Press, 1997.

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